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Fidelity and purity decay in weakly coupled composite systems

Marko Žnidarič and Tomaž Prosen

Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia

E-mail: znidaricm@fiz.uni-lj.si and prosen@fiz.uni-lj.si

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Abstract

We study the stability of unitary quantum dynamics of composite systems (for example: central system + environment) with respect to weak interaction between the two parts. Unified theoretical formalism is applied to study different physical situations: (i) coherence of a forward evolution as measured by the purity of the reduced density matrix, (ii) stability of time evolution with respect to small coupling between subsystems and (iii) Loschmidt echo measuring dynamical irreversibility. Stability has been measured either by fidelity of pure states of a composite system, or by the so-called *reduced fidelity* of reduced density matrices within a subsystem. Rigorous inequality among fidelity, reduced fidelity and purity is proved and a linear response theory is developed expressing these three quantities in terms of time correlation functions of the generator of interaction. The qualitatively different cases of regular (integrable) or mixing (chaotic in the classical limit) dynamics in each of the subsystems are discussed in detail. Theoretical results are demonstrated and confirmed in a numerical example of two coupled kicked tops.

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1. Introduction

Recently we have witnessed a strong revival of interest in the theoretical questions related to decoherence, in particular due to the immense practical potential of the upcoming quantum information processing technology [1]. In order to design quantum devices capable of coherent quantum manipulation, one has to be able to control and minimize the decoherence due to an unavoidable weak coupling between the system of the device and the environment. Traditionally, one uses idealized harmonic heat bath models of an environment, and very often also harmonic description of the central system (e.g. [2]). However, the rate of decoherence, or quantum dissipation, may depend on the intrinsic dynamics of the central system, whether chaotic or not. It has been argued by Zurek [3, 4] that the rate of decoherence as characterized by the von Neumann entropy of the reduced density matrix increases with the rate given by the classical phase space stretching rate (Lyapunov exponent), for the quantum state which is initially given as minimal uncertainty Gaussian wave packet and for sufficiently short times. Recently, Jalabert and Pastawski have studied the so-called *quantum Loschmidt echo* [5], or *quantum fidelity*, which may be treated as a measure of dynamical reversibility under a slight change in the Hamiltonian, and found a similar relation to the classical stretching rate for short times as for decoherence. This reflects correspondence between classical and quantum evolution of wave packets up to the Ehrenfest time $-\log \hbar$ [6]. Later on, numerous papers appeared addressing related issues [7, 8].

On the other hand, we have shown [9, 10] that the behaviour of quantum fidelity may be completely different, if either the time is longer than the Ehrenfest time, or the initial (pure) state is more complex, e.g. random. In general, the rate of fidelity decay is given by the time integral of the autocorrelation function of the perturbation operator, and this is bigger the less chaotic is the dynamics thus making the fidelity lower, and vice versa.

In previous papers [9–13] we have studied fidelity and the so-called purity fidelity characterizing the stability of quantum evolution, or quantum echoes, to the perturbation of dynamics in a generally coupled composite system, i.e. where both perturbed and unperturbed systems were coupled. In a situation where we are interested only in the properties of a central subsystem, one usually does not have the above general situation but a more specific one. Namely, the coupling to the environment is usually unwanted and small. So in an ideal (unperturbed) case we would like to have two decoupled systems. In that case, we are interested in how the coupling of a central system to the environment changes the properties of the central system. So rather than comparing evolution in two general systems, in which both have coupling between subsystems, we have a situation where in an unperturbed case the systems are uncoupled and become coupled only because of the perturbation. This situation is studied in the present paper. We should stress that we use the terms *central system* and *environment* merely to refer to two abstract pieces of a general composite system *without* making any of the traditional assumptions, such as infinity of degrees of freedom and/or Markovian dynamics of the environmental part. Note also that we shall always refer to the regime where the perturbation is *semiclassically small*. In the opposite regime the situation can often be understood using classical concepts only, such as exponential (Lyapunov) stretching [5].

We develop a unified theoretical framework to deal with different physical situations: (i) coherence of forward evolution as measured by purity of the reduced density matrix traced over the subsystem, (ii) stability of time evolution with respect to small coupling between the subsystems and (iii) Loschmidt echo measuring dynamical irreversibility. Stability is measured either by fidelity of pure states of a composite system, or by the so-called *reduced fidelity* of reduced density matrices within a subsystem. We find a general linear response formula expressing the fidelity, the reduced fidelity and the purity, in terms of time-correlation functions of the generator of the perturbation within the subsystems. We emphasize that the decay rates as given by linear response formalism are usually valid also in the regime of small fidelity/purity. Our general qualitative conclusion is that all three quantities decrease slower with the increasing chaoticity of the dynamics in the subsystems. Our theoretical results are clearly demonstrated on a system of two coupled quantized kicked tops [14]. In addition, we find some intriguing numerical results on algebraic long-time tails of some of the stability measures.

2. Characterizing stability and coherence of reduced time evolution

Our system consists of a central system and an environment, henceforth denoted by subscripts 's' and 'e', respectively. The Hilbert space of a composite system is a direct product $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$. We will compare quantum evolutions generated by two Hamiltonians, the unperturbed H and the perturbed H_δ,

$$H = H_s \otimes \mathbb{1}_e + \mathbb{1}_s \otimes H_e \qquad H_\delta = H + \delta \cdot V \tag{1}$$

where $H_{s,e}$ acts only on the corresponding subspace $\mathcal{H}_{s,e}$ and δ is a dimensionless coupling strength, and V is a general perturbation operator that couples both systems. We stress again that we use the names *central system* and *environment* only for the sake of convenience, and they only refer to two abstract pieces of a composite quantum system regardless of their structure. The usual measure of stability of overall unitary evolution on the total Hilbert space $\mathcal{H}$ is the pure state fidelity (equivalent to quantum Loschmidt echo), which is the overlap between perturbed $|\psi_\delta(t)\rangle = U_\delta(t)|\psi(0)\rangle$ and unperturbed $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ time evolving states, where $U(t) = e^{-iHt/\hbar}$ and $U_\delta(t) = e^{-iH_\delta t/\hbar}$ are the unitary propagators. It turns out to be useful to define an *echo operator* $M_\delta(t)$ [11, 12][1],

$$M_\delta(t) := U^\dagger(t) U_\delta(t). \tag{2}$$

Writing the density matrices,

$$\rho(t) := |\psi(t)\rangle\langle\psi(t)| = U(t)\rho(0)U^\dagger(t) \tag{3}$$

$$\rho_\delta(t) := |\psi_\delta(t)\rangle\langle\psi_\delta(t)| = U_\delta(t)\rho(0)U_\delta^\dagger(t) \tag{4}$$

$$\rho_{M\delta}(t) := M_\delta(t)\rho(0)M_\delta^\dagger(t) \tag{5}$$

the fidelity can be concisely written as[2]

$$F(t) := |\langle\psi_\delta(t)|\psi(t)\rangle|^2 = \mathrm{Tr}[\rho(0)\rho_{M\delta}(t)]. \tag{6}$$

The echo operator (2) can be rewritten as

$$M_\delta(t) = \hat{\mathcal{T}} \exp(-i\Sigma(t)\delta/\hbar) \qquad \text{with} \quad \Sigma(t) := \int_0^t V(\tau)\,d\tau \tag{7}$$

where $\hat{\mathcal{T}}$ is a time-ordering operator, and $V(t) := U^\dagger(t)VU(t)$ is the perturbation in the interaction picture. This representation of the echo operator (7) is very convenient as it can be used [12] to derive various results on the behaviour of fidelity (6) and related functions. As $\rho_{M\delta}(t)$ is nothing but the density operator of the total system in the interaction picture, it satisfies

$$\frac{d}{dt}\rho_{M\delta}(t) = \frac{i\delta}{\hbar}[\rho_{M\delta}(t), V(t)] \tag{8}$$

where $[A, B] := AB - BA$.

The fidelity is a measure of the distance within the whole Hilbert space $\mathcal{H}$. On the other hand, in the spirit of our study of a central system, we would like to have a quantity that would measure distance only within the central system space $\mathcal{H}_s$. So we should not care if the environmental states are corrupted by the perturbation, but would only like the evolution on the subspace $\mathcal{H}_s$ to be preserved as closely as possible. With this aim we define a new quantity that we call *reduced fidelity*, which is the fidelity between the reduced perturbed

[1] Note that here the order of perturbed and unperturbed propagators is interchanged with respect to [11, 12] in order to facilitate exact partial tracing, but is otherwise inessential.

[2] Note that fidelity is often defined (see e.g. [1]) as a square-root of expression (6).

$\rho_{s\delta}(t) := \mathrm{Tr}_e\,\rho_\delta(t)$ and unperturbed $\rho_s(t) := \mathrm{Tr}_e\,\rho(t) = U_s(t)\rho_s(0)U_s^\dagger(t)$ density matrices, which start from the same product (disentangled) initial state

$$|\psi(0)\rangle = |\psi(0)\rangle_s \otimes |\psi(0)\rangle_e \tag{9}$$

and we use the obvious notation $U_s(t) := \mathrm{e}^{-\mathrm{i}H_s t/\hbar}$. Note that the product form of the initial state (9) is assumed throughout this paper. The reduced fidelity F_R therefore reads

$$F_R(t) := \mathrm{Tr}_s[\rho_s(t)\rho_{s\delta}(t)] = \mathrm{Tr}_s[\rho_s(0)\,\mathrm{Tr}_e\{\rho_{M\delta}(t)\}]. \tag{10}$$

The reduced fidelity measures the distance between the two reduced density matrices. However, we stress that expression (10) agrees with the quantum-information-theoretic definition of fidelity (see e.g. [1]) between two mixed states because one of them, namely $\rho_s(t)$, is always a *pure state* due to the separability of unperturbed dynamics. $F_R(t)$ can be interpreted either as an inner product of two reduced density operators propagated by two nearby Hamiltonians, or as an inner product of the initial and the final reduced density operator after the echo dynamics.

On the other hand, if we are interested only in the separability (disentanglement) of the final echo density matrix $\rho_{M\delta}(t)$, the relevant quantity is *purity fidelity* [11, 12]

$$F_P(t) := \mathrm{Tr}_s[\mathrm{Tr}_e\,\rho_{M\delta}(t)]^2. \tag{11}$$

Now we shall use the fact that H is separable in two subsystems (1), so that purity fidelity is in this case equal to *purity* [3] of the coupled forward time evolution

$$I(t) := \mathrm{Tr}_s[\mathrm{Tr}_e\,\rho_\delta(t)]^2 = \mathrm{Tr}_s[\rho_{s\delta}(t)]^2. \tag{12}$$

In order to see that purity fidelity is equal to purity in this situation, we bring the separable propagator ($U = U_s \otimes U_e$) out of the innermost trace and use the cyclic property of a trace operation in the definition of $F_P(t)$ (11),

$$F_P(t) = \mathrm{Tr}_s\big[\,\mathrm{Tr}_e\big(U^\dagger U_\delta \rho(0) U_\delta^\dagger U\big)\big]^2 = \mathrm{Tr}_s\left[U_s^\dagger\,\mathrm{Tr}_e\left\{U_\delta\rho(0)U_\delta^\dagger\right\}U_s\right]^2 = I(t). \tag{13}$$

This can be understood as a consequence of the purity being constant during the evolution with the separable Hamiltonian H. However in the general case, the purity fidelity is a property of echo dynamics and is different from the purity of forward dynamics. Since in this paper we are interested in the former case (1), we will from now on use a symbol $I(t)$ instead of $F_P(t)$.

In summary, all three quantities, namely $F(t)$, $F_R(t)$ and $I(t)$, measure the stability of a composite system to perturbations. The fidelity $F(t)$ measures the stability of a whole state, the reduced fidelity gives the stability on $\mathcal{H}_s$ subspace, and the purity measures the separability of $\rho_\delta(t)$. One expects the fidelity to be the most restrictive quantity of the three—$\rho(t)$ and $\rho_\delta(t)$ must be similar for $F(t)$ to be high. For $F_R(t)$ to be high, only the reduced density matrices $\rho_s(t) = \mathrm{Tr}_e[\rho(t)]$ and $\rho_{s\delta}(t) = \mathrm{Tr}_e[\rho_\delta(t)]$ must be similar, and finally, purity $I(t)$ (12) is high if only $\rho_\delta(t)$ factorizes. In a previous paper [11] we proved an inequality $F^2(t) \leqslant F_P(t)$. Along the same lines one can prove the following general inequality:

$$F^2(t) \leqslant F_R^2(t) \leqslant I(t). \tag{14}$$

Proof. Write $\rho(0) = \rho_s \otimes \rho_e$. In this special case, Uhlmann's theorem (noncontractivity of fidelity) [15] states for any pure states $\rho_{M\delta}(t)$ and $\rho(0)$ that

$$F(t) = \mathrm{Tr}\,[\rho(0)\rho_{M\delta}(t)] \leqslant \mathrm{Tr}_s[\rho_s\,\mathrm{Tr}_e\{\rho_{M\delta}(t)\}] = F_R(t). \tag{15}$$

Then, squaring and applying the Cauchy–Schwartz inequality $|\mathrm{Tr}(A^\dagger B)|^2 \leqslant \mathrm{Tr}(AA^\dagger)\mathrm{Tr}(BB^\dagger)$ we obtain $F^2(t) \leqslant F_R^2(t) \leqslant \mathrm{Tr}_s[\mathrm{Tr}_e\,\rho_{M\delta}(t)]^2 = F_P(t) = I(t)$ with equality being satisfied only in the trivial case of $M_\delta(t) \equiv \mathbb{1}$, i.e. when $\delta = 0$. □

3. Linear response

Next we proceed by expanding all three important quantities in powers of the perturbation strength δ. Although this procedure is very elementary, it greatly helps in understanding the behaviour of various measures of stability. The lowest order is given by various two-point time-correlation functions of the perturbation and the decay time scale can be inferred from the behaviour of time integrals of these correlations. What is more, the dependence of this time scale on δ and $\hbar$ as well as on the dynamics of a system (fast correlation decay or absence of correlation decay) is explicit. We start by expanding the echo operator

$$M_\delta(t) = \mathbb{1} - \frac{\mathrm{i}\delta}{\hbar}\Sigma(t) - \frac{\delta^2}{2\hbar^2}\hat{\mathcal{T}}\Sigma^2(t) + \cdots . \tag{16}$$

The leading order expansion of fidelity (6) is then

$$F(t) = 1 - \left(\frac{\delta}{\hbar}\right)^2 C(t) \qquad C(t) := \langle\Sigma^2(t)\rangle - \langle\Sigma(t)\rangle^2 \tag{17}$$

where $\langle\bullet\rangle$ denotes an expectation in the product initial state $|\psi(0)\rangle = |1, 1\rangle$, with the general notation for a complete basis of Hilbert space $|i, \nu\rangle := |i\rangle_{\mathrm{s}} \otimes |\nu\rangle_{\mathrm{e}}$. If explicitly written, the coefficient $C(t)$ is only an *integral of autocorrelation function* and reads

$$C(t) = \int_0^t \int_0^t \{\langle V(\xi)V(\zeta)\rangle - \langle V(\xi)\rangle\langle V(\zeta)\rangle\}\,\mathrm{d}\xi\,\mathrm{d}\zeta . \tag{18}$$

Similarly, for the reduced fidelity $F_{\mathrm{R}}(t)$ we obtain

$$\begin{aligned} F_{\mathrm{R}}(t) &= 1 - \left(\frac{\delta}{\hbar}\right)^2 \{C(t) - D(t)\} \\ D(t) &:= \langle\Sigma(t)(\rho_{\mathrm{s}} \otimes \mathbb{1}_{\mathrm{e}})\Sigma(t)\rangle - \langle\Sigma(t)\rangle^2 = \sum_{\nu\neq 1} |\langle 1, \nu|\Sigma(t)|1, 1\rangle|^2 \end{aligned} \tag{19}$$

and for the purity $I(t)$

$$\begin{aligned} I(t) &= 1 - 2\left(\frac{\delta}{\hbar}\right)^2 \{C(t) - D(t) - E(t)\} \\ E(t) &:= \langle\Sigma(t)(\mathbb{1}_{\mathrm{s}} \otimes \rho_{\mathrm{e}})\Sigma(t)\rangle - \langle\Sigma(t)\rangle^2 = \sum_{i\neq 1} |\langle i, 1|\Sigma(t)|1, 1\rangle|^2 . \end{aligned} \tag{20}$$

So far we have not specified any particular form of the perturbation V. To facilitate calculations, we now assume the simplest and physically well-justified *product* form of the interaction

$$V := V_{\mathrm{s}} \otimes V_{\mathrm{e}} . \tag{21}$$

This is a very natural choice in the studies of decoherence. Such is the usual case where one writes $V = x_{\mathrm{s}} \otimes F_{\mathrm{e}}$ meaning the coupling of *position* times *force*. Henceforth, the operator V_{e} will be referred to as 'force' and V_{s} as 'position'. Following this assumption, the three coefficients $C(t)$, $D(t)$ and $E(t)$ can be written explicitly in terms of separate correlation functions over different spaces, namely $\langle\bullet\rangle_{\mathrm{e,s}} = \langle 1|_{\mathrm{e,s}} \bullet |1\rangle_{\mathrm{e,s}}$:

$$\begin{aligned} C(t) &= \int_0^t \int_0^t \{\langle V_{\mathrm{s}}(\xi)V_{\mathrm{s}}(\zeta)\rangle_{\mathrm{s}}\langle V_{\mathrm{e}}(\xi)V_{\mathrm{e}}(\zeta)\rangle_{\mathrm{e}} - \langle V_{\mathrm{s}}(\xi)\rangle_{\mathrm{s}}\langle V_{\mathrm{s}}(\zeta)\rangle_{\mathrm{s}}\langle V_{\mathrm{e}}(\xi)\rangle_{\mathrm{e}}\langle V_{\mathrm{e}}(\zeta)\rangle_{\mathrm{e}}\}\,\mathrm{d}\xi\,\mathrm{d}\zeta \\ D(t) &= \int_0^t \int_0^t \{\langle V_{\mathrm{s}}(\xi)\rangle_{\mathrm{s}}\langle V_{\mathrm{s}}(\zeta)\rangle_{\mathrm{s}}[\langle V_{\mathrm{e}}(\xi)V_{\mathrm{e}}(\zeta)\rangle_{\mathrm{e}} - \langle V_{\mathrm{e}}(\xi)\rangle_{\mathrm{e}}\langle V_{\mathrm{e}}(\zeta)\rangle_{\mathrm{e}}]\}\,\mathrm{d}\xi\,\mathrm{d}\zeta \\ E(t) &= \int_0^t \int_0^t \{[\langle V_{\mathrm{s}}(\xi)V_{\mathrm{s}}(\zeta)\rangle_{\mathrm{s}} - \langle V_{\mathrm{s}}(\xi)\rangle_{\mathrm{s}}\langle V_{\mathrm{s}}(\zeta)\rangle_{\mathrm{s}}]\langle V_{\mathrm{e}}(\xi)\rangle_{\mathrm{e}}\langle V_{\mathrm{e}}(\zeta)\rangle_{\mathrm{e}}\}\,\mathrm{d}\xi\,\mathrm{d}\zeta . \end{aligned} \tag{22}$$

The above correlation integrals (22) are the starting point for our theoretical investigations. In certain situations they can be simplified even further. We will study four different regimes in which simplification is possible: (i) mixing regime (corresponding to chaotic classical dynamics in both subspaces) in which arbitrary correlation functions appearing in (22) decay to 0 and their integrals thus grow as $\propto t$, (ii) regular regime in which due to absence of mixing the whole correlation integral $C(t)$ (or $D(t)$, or $E(t)$) grows as $\propto t^2$, and two regimes in which we have a separation of time scales, with the time scale of the environment being much shorter than the time scale of the central system. In this case, we will work out two different regimes depending on the mixing property of the environment, namely (iii) 'fast mixing' environment where environmental correlations decay, and (iv) 'fast regular' environment where the environmental correlation function has a non-vanishing time average value. The decay of fidelity and purity fidelity in mixing and regular regimes has already been discussed extensively [9–12]. In this work we discuss not only a new quantity, namely the reduced fidelity, but also the unperturbed dynamics that is now separable and thus not ergodic on the total space.

It is interesting to note that the general inequality (14) is clearly satisfied in our linear response results since the functions $E(t)$ and $D(t)$ are written in terms of sums of non-negative real numbers (equations (19) and (20)) and hence they themselves are always non-negative.

We want to stress that all the results of this section can be directly translated to the discrete time case of quantum maps (kicked quantum systems) by simply treating t as an integer variable and replacing all the integrals by sums, $\int_0^t \to \sum_0^{t-1}$.

4. Numerical experiment: two coupled kicked tops

We will now illuminate and demonstrate our theoretical predictions with a numerical example of two coupled kicked tops. In addition, numerical simulations will provide us with some insight into the asymptotic behaviour for long times and small fidelity/purity. Note that here the time is discrete (integer) and measured in the number of kicking periods (steps). A single kicked top [16] has a unitary one step (Floquet) propagator

$$U(\alpha,\gamma) = \exp(-\mathrm{i}\gamma J_{\mathrm{y}}) \exp\left(-\mathrm{i}\alpha J_{\mathrm{z}}^2 \big/ 2J\right). \tag{23}$$

The level of chaoticity (e.g. the rate of mixing) of a single top can be varied by varying α and the time scale can be influenced by changing the angle γ. In order to be able to study the reduced fidelity and the purity we couple two kicked tops, where one top will act as an 'environment' and the other as a 'central system'. Two coupled kicked tops [10, 14] have the following unitary one step propagator,

$$U_\delta := U_{\mathrm{s}}(\alpha_{\mathrm{s}},\gamma_{\mathrm{s}}) U_{\mathrm{e}}(\alpha_{\mathrm{e}},\gamma_{\mathrm{e}}) \exp(-\mathrm{i}\delta \cdot V/\hbar) \tag{24}$$

with $U_{\mathrm{s,e}}$ being propagators for a single kicked top of a system or environment. The unperturbed propagator $U := U_{\delta=0}$ is simply obtained by putting $\delta = 0$ into expression (24). Perturbation $V = V_{\mathrm{s}} \otimes V_{\mathrm{e}}$ is chosen to be of two different forms: for regimes (i), (ii) and (iii) both $V_{\mathrm{s,e}} = J_{\mathrm{z}}/J$ have the same form, whereas in the regime (iv) we take $V_{\mathrm{s}} = J_{\mathrm{z}}/J$ and $V_{\mathrm{e}} = J_{\mathrm{z}}^2 \big/ J^2$. The reason for choosing a different form of perturbation in case (iv) is that we want the environmental time-correlation function to have a non-vanishing time average in order to yield generic results. In the other cases (i)–(iii) the precise form of the force and position operators is irrelevant, so $V = J_{\mathrm{z}}/J$ provides the simplest choice. The Planck constant is determined by J as $\hbar = 1/J$, so that the semiclassical limit implies $J \to \infty$. The initial condition will always be a direct product of coherent states for both tops, $|1, 1\rangle = |\varphi_s, \vartheta_s\rangle \otimes |\varphi_e, \vartheta_e\rangle$, with the expansion of the $SU(2)$ coherent state $|\varphi, \vartheta\rangle$ into eigenbasis $|m\rangle$ of J_z being

$$|\varphi, \vartheta\rangle = \sum_{m=-J}^{J} \binom{2J}{J+m}^{1/2} \cos(\vartheta/2)^{J+m} \sin(\vartheta/2)^{J-m} \, e^{-im\varphi} |m\rangle. \tag{25}$$

Let us now work out the details of the fidelity, reduced fidelity and purity decay in all four regimes.

4.1. Mixing regime

If the combined correlation function decays to zero sufficiently fast, so that its time integral converges and $C(t) \propto t$ (18), we can define a kind of transport coefficient

$$\sigma := \lim_{t\to\infty} C(t)/2t. \tag{26}$$

The linear response formalism thus gives us initial linear decrease of fidelity. On the other hand the coefficient $D(t) + E(t)$, occurring in the expression for purity, is small compared to $C(t)$ [11], namely $\{D(t) + E(t)\}/C(t) \sim 1/N_s + 1/N_e$ ($N_{s,e}$ are dimensions of $\mathcal{H}_{s,e}$) and the same holds for each individual $E(t)$ or $D(t)$. Therefore, up to a semiclassically vanishing correction, all three quantities are expected to decay on the same time scale. What is more, if multi-time-correlation functions also fall off fast [9–11], then the shape of decay for longer times is exponential with the exponent given by the linear response formula,

$$F^2(t) = F_R^2(t) = I(t) = e^{-2t/\tau_m} \qquad \tau_m = \frac{\hbar^2}{2\delta^2\sigma}. \tag{27}$$

This formula is expected to be valid for times longer than the Ehrenfest time or classical mixing time, depending on the initial coherent or random state, respectively, and for sufficiently small δ such that the value of fidelity/purity is still far above the saturation for the above-mentioned time scale (see [10] for a detailed discussion). Therefore, if the central system and the environment are both separately mixing (by mixing we mean decay of all time-correlation functions), the fidelity, reduced fidelity and purity decay in the same way. This means that the decay of purity, decay of reduced fidelity and decay of fidelity have the same principal mechanism, that is 'corruption' of the state vector as a whole which gives a dominant contribution over more subtle issues of destruction of coherence. Note that stronger mixing (usually connected to stronger chaoticity of a classical system) means smaller correlation integral σ and thus slower decay of purity fidelity and reduced fidelity. This is totally opposite to the very short time behaviour described in [3, 4, 14].

For numerical verification of this result we chose the perturbation $V_{s,e} = J_z/J$ and parameters $\alpha_{s,e} = 30$, $\gamma_{s,e} = \pi/2.1$ and $J = 1/\hbar = 200$ giving the effective size of the Planck constant. The coherent initial state is placed at $\vartheta_{s,e} = \pi/\sqrt{3}$, $\varphi_{s,e} = \pi/\sqrt{2}$. In figure 1 we plot the time dependence of correlation integrals occurring in the expressions for fidelity (17), reduced fidelity (19) and purity (20), showing that the terms $D(t)$ and $E(t)$ are really negligible as all three quantities shown are practically equal for times longer than the Ehrenfest time. Next we show in figure 2 the decay of fidelity $F(t)$, reduced fidelity $F_R(t)$ and purity $I(t)$ for the same parameters and for the perturbation strength $\delta = 8 \times 10^{-4}$. Clean exponential decay is observed in all three cases, on a time scale τ_m (27) given exactly by the lowest order linear response expression (17) in terms of σ (26) which is obtained from the data of figure 1. Exponential decay, of course, persists only up to the saturation value determined by a finite Hilbert space size [10].

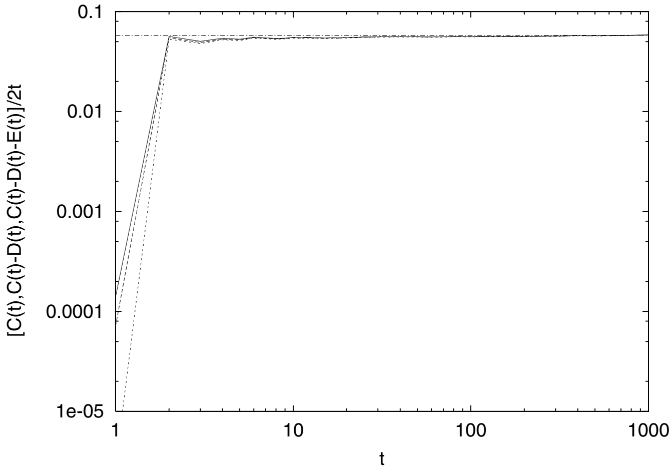

Figure 1. Correlation sums (22) in the mixing (chaotic) regime (see text) divided by $2t$: $C(t)$ (solid), $C(t) - D(t)$ (long dashed), $C(t) - D(t) - E(t)$ (short dashed). The horizontal (chain) line shows the best fitting value of σ (26).

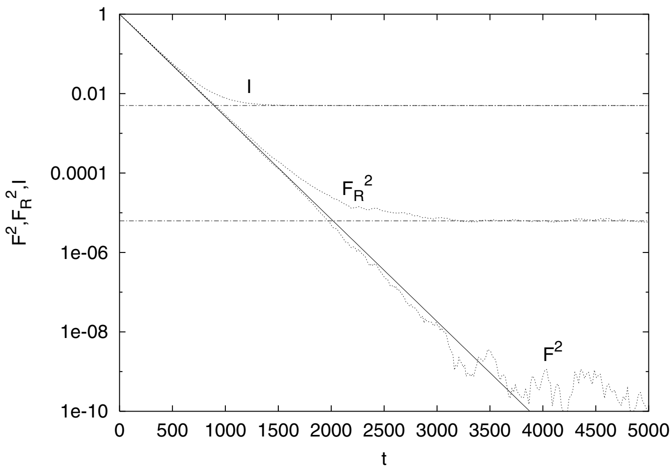

Figure 2. Decay of $F^2(t)$, $F_{\rm R}^2(t)$ and $I(t)$ (dotted curves) in the mixing (chaotic) regime. The solid line gives exponential decay (27) with $\tau_{\rm m}$ calculated from σ in figure 1. Horizontal chain lines give the saturation values of purity and reduced fidelity, $1/200$ and $1/400^2$, respectively.

4.2. Regular regime

If the system is regular as a whole, then the time-correlation function will generally not decay to zero, but will have a non-vanishing average value

$$\bar{c}_{\rm F} := \lim_{t\to\infty} C(t)/t^2. \tag{28}$$

Similar coefficients can be defined for the average of $C(t) - D(t) \asymp \bar{c}_{\rm R} t^2$ occurring in the expansion of the reduced fidelity, and $C(t) - D(t) - E(t) \asymp \bar{c}_{\rm I} t^2$ for the purity. Note that $C(t)$ is proportional to $\hbar$ for coherent initial states. As shown before [11], the expression $D(t)+E(t)$ is almost equal to $C(t)$ for coherent initial states in the regime of regular (integrable) classical dynamics. The expression $C(t) - D(t) - E(t)$ occurring in the formula for purity (20) is therefore of the order $\hbar^2$ and the decay time scale for purity decay is $\hbar$ independent. This cancellation in the leading order in $\hbar$ happens due to both $E(t)$ and $D(t)$ terms and the reduced fidelity will therefore decay on approximately the same time scale as the fidelity. For coherent initial states, one can show [10, 11] that the shape of decay is a Gaussian

$$F(t) = \mathrm{e}^{-(t/\tau_\mathrm{r})^2} \qquad \tau_\mathrm{r} = \frac{\hbar}{\delta\sqrt{\bar{c}_\mathrm{F}}} \tag{29}$$

and with a similar expression, only with $\bar{c}_\mathrm{R}$ replacing $\bar{c}_\mathrm{F}$, for reduced fidelity. Remember that for coherent initial states $\bar{c}_\mathrm{F,R} \propto \hbar$ and therefore $\tau_\mathrm{r} \propto \sqrt{\hbar}$. The time scale for decay of purity is again given by an analogous expression, namely $\hbar/\delta\sqrt{\bar{c}_\mathrm{I}}$, with $\bar{c}_\mathrm{I} \propto \hbar^2$, but as we will see in the numerical simulations and discuss later, its long time behaviour is not a Gaussian but has an algebraic tail instead.

For the numerical demonstration we take two kicked tops with $V_\mathrm{s,e} = J_\mathrm{z}/J$, $J = 200$, $\gamma_\mathrm{s,e} = \pi/2.1$ and $\alpha_\mathrm{s,e} = 0$ in order to have regular dynamics in both subspaces. The initial coherent state is placed at $(\vartheta, \varphi)_\mathrm{s} = (\pi/\sqrt{3}, \pi/\sqrt{2})$ and $(\vartheta, \varphi)_\mathrm{e} = (\pi/\sqrt{3}, 3\pi/\sqrt{7})$. In fact, the dynamics for $\gamma_\mathrm{e,s} = \pi/2$ (approximating a 'more generic' value $\gamma = \pi/2.1$, and $\alpha = 0$) is very simple, namely it is a pure rotation by an angle $\pi/2$, and all three correlation integrals (sums) can be calculated explicitly. Coherent state expectation values of the time-averaged correlation functions are then easily evaluated using formulae for the expectation values of the lowest powers of $J_\mathrm{x}, J_\mathrm{y}, J_\mathrm{z}$. If we denote with $\mathbf{n}_\mathrm{s,e} = (x, y, z)_\mathrm{s,e} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)_\mathrm{s,e}$ the unit vectors in the directions of the initial coherent states of the central system and the environment, then the results for $\bar{c}$ (for this special case of $\pi/2$ rotation with $\alpha = 0$) are

$$\begin{aligned}
\bar{c}_\mathrm{F} &= \frac{1}{8J}\left[2 - y_\mathrm{s}^2 - y_\mathrm{e}^2 - 2(\mathbf{n}_\mathrm{s}\cdot\mathbf{n}_\mathrm{e} - y_\mathrm{s}y_\mathrm{e})^2\right] + \frac{1}{16J^2}[(y_\mathrm{s} - y_\mathrm{e})^2 + (\mathbf{n}_\mathrm{s}\cdot\mathbf{n}_\mathrm{e} - y_\mathrm{s}y_\mathrm{e})^2] \\
\bar{c}_\mathrm{R} &= \frac{1}{8J}\left[1 - y_\mathrm{e}^2 - (\mathbf{n}_\mathrm{s}\cdot\mathbf{n}_\mathrm{e} - y_\mathrm{s}y_\mathrm{e})^2\right] + \frac{1}{16J^2}[(y_\mathrm{s} - y_\mathrm{e})^2 + (\mathbf{n}_\mathrm{s}\cdot\mathbf{n}_\mathrm{e} - y_\mathrm{s}y_\mathrm{e})^2] \\
\bar{c}_\mathrm{I} &= \frac{1}{16J^2}[(y_\mathrm{s} - y_\mathrm{e})^2 + (\mathbf{n}_\mathrm{s}\cdot\mathbf{n}_\mathrm{e} - y_\mathrm{s}y_\mathrm{e})^2].
\end{aligned} \tag{30}$$

Note that all $\bar{c}$ are expressed in terms of the invariants of motion as they should be by definition (however, this does not mean that the perturbation itself is an invariant of motion). Here one can also see explicitly the cancellation of terms for purity, namely the $\bar{c}_\mathrm{F}$ and $\bar{c}_\mathrm{R}$ are proportional to $\hbar$, while $\bar{c}_\mathrm{I}$ is proportional to $\hbar^2$. We should note that for special positions of initial packets, the average correlator may vanish at $\bar{c} = 0$, and there the decay may be much slower and not Gaussian at all. The zeros of $\bar{c}$ are therefore very special points denoting wave packets that are very stable against perturbations. For fidelity and reduced fidelity, this slow decay can give rise to a power-law decay of average fidelity (fidelity averaged over the whole phase space).

In figure 3 we can see that the three theoretical coefficients $\bar{c}$ (30) for $\gamma_\mathrm{s,e} = \pi/2$ approximate very well the numerical calculation of correlation sums in the case of $\gamma_\mathrm{s,e} = \pi/2.1$. In figure 4 we show the decay of fidelity, reduced fidelity and purity for $\delta = 5\times 10^{-3}$. The decay of fidelity and reduced fidelity is Gaussian (29) with the decay times τ_r given very accurately by the theoretically calculated $\bar{c}_\mathrm{F}$ and $\bar{c}_\mathrm{R}$ shown with solid curves in figure 3. The decay of purity on the other hand is not Gaussian for long times. Of course, it decays quadratically as given by a linear response formula (20) for times short enough so that the purity is close to 1, but for larger times it decays with a power-law. Still, up to a constant numerical factor (independent of $\delta, \hbar$ etc), the effective purity decay time is given by a linear response formula for τ_r (29). This can be observed in figure 5 where we show numerically calculated values of τ_r based on $1/\mathrm{e} \approx 0.37$ level fidelity (purity) together with the theoretical

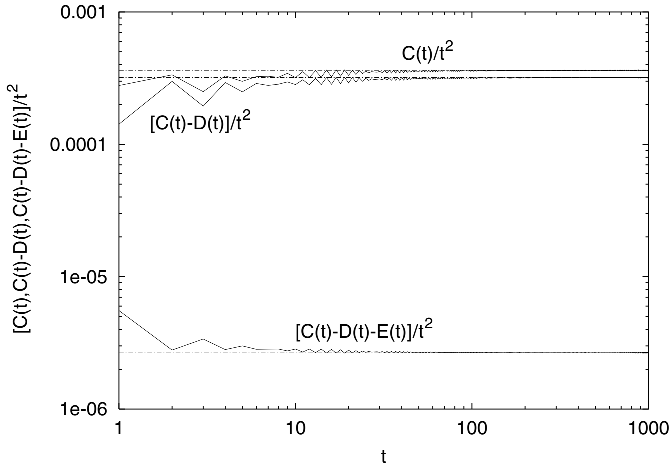

Figure 3. Correlation sums (22) in the regular regime (solid curves). Chain lines show the theoretical values of $\bar{c}_{\mathrm{F,R,I}}$ (30). See text for details.

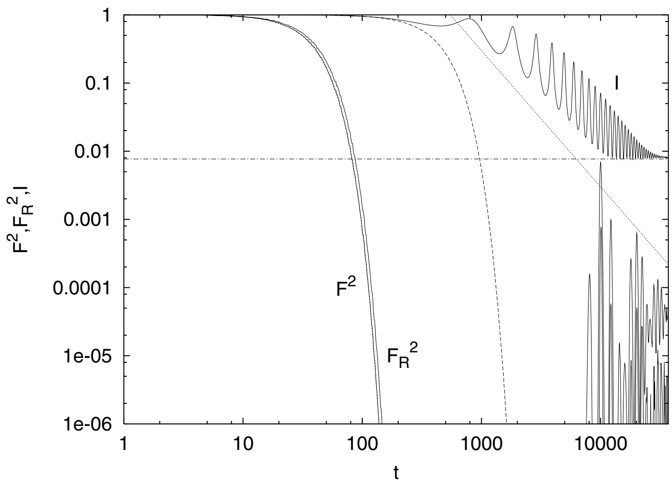

Figure 4. Decay of $F^2(t)$, $F_{\mathrm{R}}^2(t)$ and $I(t)$ (solid curves) in a regular regime (see text). The dashed curve is a Gaussian with the exponent given by $\bar{c}_{\mathrm{I}}$. Note that the theoretical Gaussians for $F(t)$ and $F_{\mathrm{R}}(t)$ are indistinguishable from the data! Dotted line has a slope -2, and the horizontal chain line gives the saturation value of $I(t)$ at $\approx 1/130$.

prediction on a basis of average correlation function ($\bar{c}$). Since the purity decay does not follow a Gaussian model the numerical and theoretical scales deviate by a numerical factor, which is however constant (independent of $\hbar$ and δ). For larger perturbation strength δ, the decay rates become less and less sensitive to perturbations in agreement with the expected saturation [5, 7]. We stress that for short times, namely for $\delta t \ll 1$, the purity of a Gaussian packet for regular dynamics can be written as an algebraic function in terms of determinants [11]. This dependence is completely independent of $\hbar$ and is a function of the scaling variable δt alone. Furthermore, we expect periodic revivals of purity with a classical beating frequency $\nu \sim 1/\delta$, again independent of $\hbar$. Based on numerical results (extending the analytical results

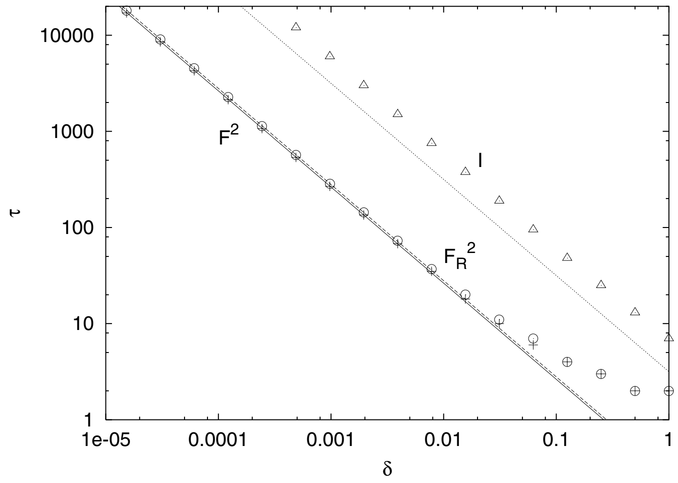

Figure 5. Times τ at which $F^2(t)$, $F_R^2(t)$, $I(t)$ fall to level 0.37 for different δ in a regular regime. Symbols—pluses, circles and triangles—are numerics whereas lines—solid, dashed, dotted—give theoretical dependence of τ_r, for $F^2(t)$, $F_R^2(t)$ and $I(t)$, respectively. All parameters are the same as for figures 3 and 4, except that here $J = 100$.

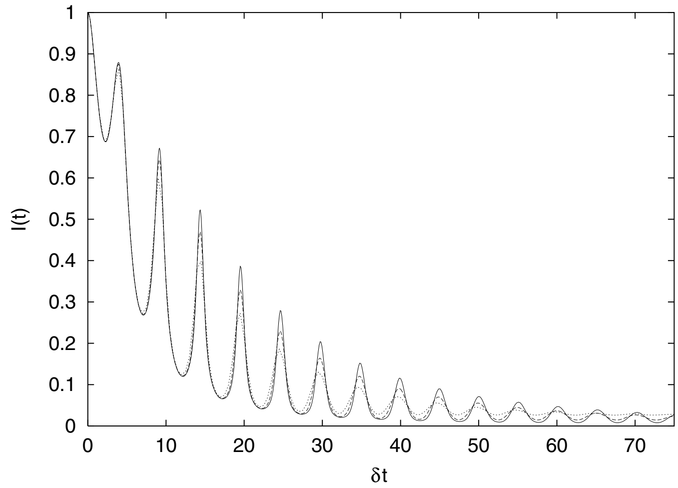

Figure 6. Decay of purity $I(t)$ in a regular regime for $J = 200$ (full curve), $J = 100$ (dashed) and $J = 50$ (dotted) and the same parameters as in figure 4. Note that $I(t)$ is almost fully independent of $\hbar$, and that curves for other values of δ completely overlap the existing curves.

for longer times) we conjecture that the overall dependence of $I(t)$ is $\hbar$ (or J) independent and its functional form can be described in terms of a scaling variable δt, namely $I = f(\delta t)$. This is demonstrated in figure 6.

The regular regime discussed above has practical importance for the emergence of the macroworld [17]. If we have a macroscopic superposition of states the decoherence time will be smaller than any dynamical time scale in a system and one is trivially in a regular regime with non-decaying correlations.

5. Separation of time scales: fast environment

The expressions for $C(t)$, $D(t)$ and $E(t)$ can be further simplified if the decay time scale of the environmental correlations $\langle V_\mathrm{e}(t)V_\mathrm{e}(t')\rangle_\mathrm{e}$ is much smaller than the time scale of the systems' correlations $\langle V_\mathrm{s}(t)V_\mathrm{s}(t')\rangle_\mathrm{s}$. The time averaging over the fast environmental part of the perturbation V_e can be performed in this case. Regarding the environmental correlation function two extreme situations are possible. Namely, the correlations of the environment decay ('fast mixing environment') so that we have a finite integral of the environmental correlation function, or the correlations of the environment do not decay ('fast regular environment') and we have a generically non-vanishing average correlation function of the environment.

5.1. Fast mixing environment

The situation, when the time scale t_e on which correlation functions for the environment decay is much smaller than the time scale t_s of the central system, is of considerable physical interest. This includes various 'Brownian'-like baths, where correlation times are smaller than the dynamical times of the system in question. The expressions for $C(t)$, $D(t)$ and $E(t)$ (22) can be significantly simplified in such a situation. We will furthermore assume that $\overline{\langle V_\mathrm{e}\rangle}_\mathrm{e} = 0$, with $\overline{\langle A\rangle} = \lim_{t\to\infty} t^{-1}\int_0^t \langle A(\xi)\rangle\,\mathrm{d}\xi$ denoting the time average, which is true if we are in an equilibrium situation (average 'force' V_e vanishes). The integration over the fast variable V_e in (22) can then be carried out and we obtain

$$\begin{aligned} C(t) &= 2t\sigma_\mathrm{e}\overline{\left\langle V_\mathrm{s}^2\right\rangle}_\mathrm{s} \\ C(t)-D(t) &= 2t\sigma_\mathrm{e}\left\{\overline{\left\langle V_\mathrm{s}^2\right\rangle}_\mathrm{s} - \overline{\langle V_\mathrm{s}\rangle^2}_\mathrm{s}\right\} \\ C(t)-D(t)-E(t) &= 2t\sigma_\mathrm{e}\left\{\overline{\left\langle V_\mathrm{s}^2\right\rangle}_\mathrm{s} - \overline{\langle V_\mathrm{s}\rangle^2}_\mathrm{s}\right\} \end{aligned} \qquad (31)$$

with

$$\sigma_\mathrm{e} := \lim_{t\to\infty}\left\langle \Sigma_\mathrm{e}^2(t)\right\rangle_\mathrm{e}\big/2t \qquad \text{and} \qquad \Sigma_\mathrm{e}(t) = \int_0^t V_\mathrm{e}(\xi)\,\mathrm{d}\xi \qquad (32)$$

being the integral of the autocorrelation function for the environmental part of perturbation V_e alone. From expressions (31) and the lowest order expansions (17), (19) and (20) we can see that the decay time scale depends only on the time average diagonal correlations of the central system $\langle V_\mathrm{s}(t)^2\rangle$ and not on the full correlation function. This is a simple consequence of the separation of time scales and means that the decay of all three stability measures does not depend on the dynamics of the central system (e.g. being mixing (chaotic) or regular.) Furthermore, the reduced fidelity $F_\mathrm{R}(t)$ and the purity $I(t)$ will decay on the same time scale (31), meaning that the decay of reduced fidelity is predominantly caused by the loss of coherence, i.e. entanglement between the two factor spaces. This means that the reduced fidelity, which is a property of echo dynamics of comparison of two slightly different Hamiltonian evolutions, is equivalent to the decay of purity or growth of linear entropy of an individual weakly coupled system.

If the initial state of a central system $\rho_\mathrm{s}(0)$ is a Gaussian wave packet (coherent state) then the dispersion of $\overline{\left\langle V_\mathrm{s}^2\right\rangle}_\mathrm{s} - \overline{\langle V_\mathrm{s}\rangle^2}_\mathrm{s}$ is a factor of order $1/\hbar$ *smaller* than $\overline{\left\langle V_\mathrm{s}^2\right\rangle}_\mathrm{s}$. Thus for coherent initial states of a central system, no matter what the initial state of the environment is, the $F_\mathrm{R}(t)$ and $I(t)$ decay on a $1/\hbar$ times *longer* time scale than $F(t)$. We have therefore reached a general conclusion based on very weak assumptions of chaotic fast environment, namely that the coherent states are most robust against decoherence (provided $t_\mathrm{e} \ll t_\mathrm{s}$), and that decoherence takes place at times longer than the correlation time of the environment $t_e \ll t_{dec}$. If decoherence is even faster than the time scale of the environment, as is the case for macroscopic superpositions, then formulae (31) are no longer valid as one is effectively in a regular regime of the previous section. Decoherence time is then independent not only of system dynamics but also of environmental dynamics characterized by σ_e (see [17]).

In the regime of a fast chaotic environment, one can immediately derive a master equation for a reduced density matrix of a central system [18, 19]. We take partial trace over the environment of expansion for $\rho_{M\delta}(t)$ (5) and (16) and write it for a small time step Δt. This time step Δt must be larger than the correlation time t_e of the environment and at the same time smaller than the correlation time t_s of the system. For the environmental part of the correlation function, we assume fast exponential decay (a particular exponential form is not essential) which is independent of the state ρ,

$$\mathrm{Tr}_e\{V_e(t)V_e(t')\rho\} \longrightarrow \frac{\sigma_e}{t_e}\exp\{-|t-t'|/t_e\}\ \mathrm{Tr}_e\,\rho. \tag{33}$$

Assuming the perturbation to be a product $V(t) = V_s(t) \otimes V_e(t)$ and the average 'force' $\mathrm{Tr}_e(V_e(t)\rho)$ to vanish together with the exponential decay of environmental correlations of the form (33) for an arbitrary state, yields a master equation for the reduced density matrix $\rho_{Ms}(t) := \mathrm{Tr}_e\,\rho_{M\delta}(t)$,

$$\dot{\rho}_{Ms}(t) = -\frac{\delta^2\sigma_e}{\hbar^2}[V_s(t), [V_s(t), \rho_{Ms}(t)]]. \tag{34}$$

For numerical demonstration we choose $V_{s,e} = J_z/J$, $J = 200$, $\delta = 1.5\times10^{-3}$, coherent initial state at $(\vartheta,\varphi)_{s,e} = (\pi/\sqrt{3}, \pi/\sqrt{2})$ and parameters $\alpha_s = 0$, $\gamma_s = \pi/50$ for the central system and $\alpha_e = 30$, $\gamma_e = \pi/2.1$ for the environment. Actually, we could take any value of α_s and would get qualitatively similar results. The only advantage of using regular central dynamics $\alpha_s = 0$ is that it is then possible to explicitly calculate averages $\overline{\langle V_s^2\rangle_s}$ and $\overline{\langle V_s^2\rangle_s} - \overline{\langle V_s\rangle^2}_s$. Namely, if $\alpha_s = 0$ and $\gamma_s \ll 1$ we obtain

$$\begin{aligned}\overline{\langle V_s^2(t)\rangle_s} &= \frac{1}{2}\left(1-y_s^2\right) + \frac{1}{4J}\left(1+y_s^2\right)\\ \overline{\langle V_s^2\rangle_s} - \overline{\langle V_s\rangle^2}_s &= \frac{1}{4J}\left(1+y_s^2\right).\end{aligned} \tag{35}$$

The values of these two quantities for our initial condition are shown in figure 7 with two dotted lines (by pure coincidence we have $\sigma_e \approx \overline{\langle V_s^2(t)\rangle_s}$), together with numerically calculated time-dependent (not yet averaged) $\langle V_s^2(t)\rangle_s$ and $\langle V_s^2(t)\rangle_s - \langle V_s(t)\rangle_s^2$ for our choice of $\gamma_s = \pi/50$. These time-dependent values oscillate on a time scale ≈ 50, which is much longer than the time ≈ 10 in which σ_e (32) converges and so the assumption $t_e \ll t_s$ is justifiable. The value of all these three quantities is then used in linear response formulae (31) to give us time scales on which F, F_R and I decay. The results are shown in figure 8. We can see that the fidelity again decays exponentially, but the reduced fidelity and purity have power-law-like tails. They decay on a time scale still roughly given by the lowest order expansions (31) and the values of σ_e (numerical from figure 7) and $\overline{\langle V_s^2\rangle_s} - \overline{\langle V_s\rangle^2}_s$ (theoretical expression (35)) which can be seen in figure 9. The same general conclusion again holds: the more chaotic the environment is (the smaller σ_e), the slower the decay of all three quantities. Purity and reduced fidelity both decay on a $1/\hbar$ longer time scale than the fidelity in accordance with expressions (35) for coherent initial states.

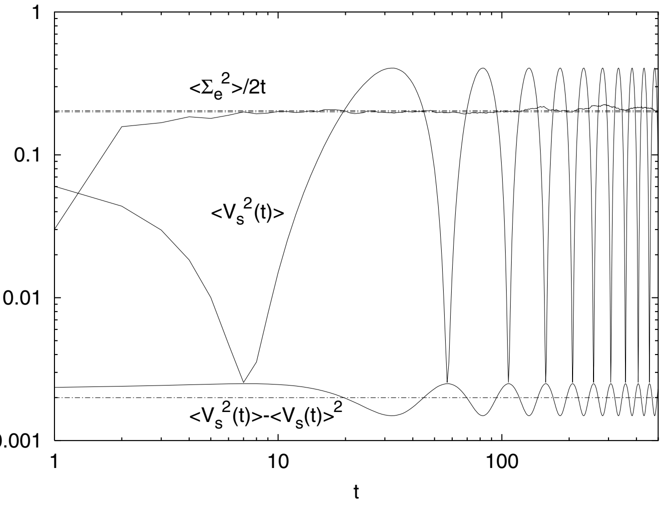

Figure 7. Various correlation sums from formulae (31) in a fast chaotic regime (solid curves, as indicated in the figure). Chain lines indicate corresponding theoretical time averages. For details see text.

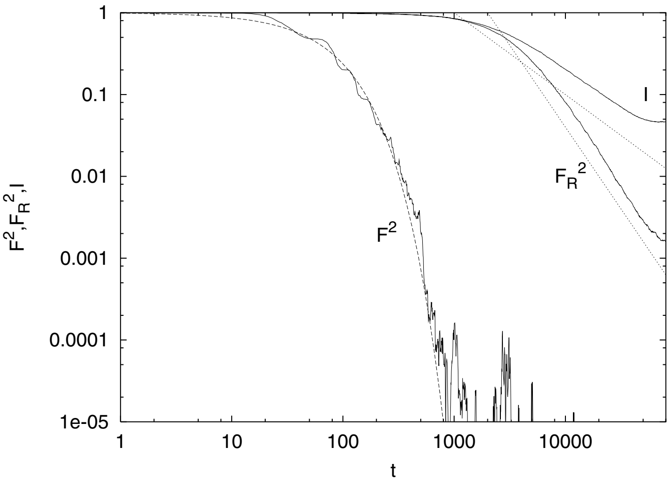

Figure 8. Decay of $F^2(t)$, $F_R^2(t)$ and $I(t)$ for a fast chaotic environment. The dashed line is exponential with the exponent given by the values of σ_e and $\overline{\langle V_s^2\rangle_s}$ (31) and the two dotted lines have slopes -2 and -1.

5.2. Fast regular environment

Here we will explore a perhaps less physical situation of a regular environmental dynamics. For a regular environment, the double integral of environment correlations grows $\propto t^2$ due to non-decay (plateau) of correlation function and we can define the average correlation function

$$\overline{c}_e := \lim_{t\to\infty} \left\langle \Sigma_e^2(t)\right\rangle_e \big/ t^2. \tag{36}$$

If in addition the correlations of the system also do not decay then the correlation sum of the total system will grow as $\propto t^2$ which is a regular regime discussed previously. Here, we will

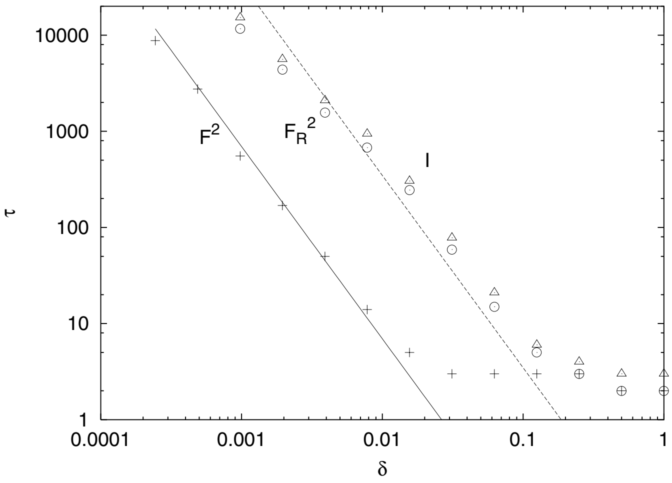

Figure 9. Times τ at which $F^2(t)$, $F_R^2(t)$, $I(t)$ fall to level 0.37 for different δ and fast chaotic environment. Symbols give numerics and lines give theoretical dependence of τ (same as in figure 5). All for $J = 100$.

focus on a different situation where the integral of the system's correlation function converges, $C_s(t) \propto t$, i.e. the central dynamics is mixing (chaotic). We will additionally assume the average 'position' V_s to be zero, $\overline{\langle V_s \rangle_s} = 0$. The transport coefficient of a system σ_s is then

$$\sigma_s := \lim_{t\to\infty} \left\langle \Sigma_s^2(t) \right\rangle_s \big/ 2t \qquad \Sigma_s(t) = \int_0^t V_s(\xi)\,d\xi. \tag{37}$$

The expressions for $C(t)$, $D(t)$ and $E(t)$ (22) for the present case can be simplified to

$$\begin{aligned} C(t) &= 2t\sigma_s\bar{c}_e \\ C(t) - D(t) &= 2t\sigma_s\bar{c}_e \\ C(t) - D(t) - E(t) &= 2t\sigma_s \left\{ \bar{c}_e - \overline{\langle V_e \rangle_e^2} \right\}. \end{aligned} \tag{38}$$

An important thing we note immediately is that now the reduced fidelity $F_R(t)$ decays on the same time scale as fidelity $F(t)$. This must be contrasted with the case of a fast mixing environment (31), where $F_R(t)$ decayed on the same time scale as the purity. If the initial state of the environment $\rho_e(0)$ is a coherent state, then the purity will decay on a $1/\hbar$ times longer time scale than fidelity and reduced fidelity. On the other hand, for a random initial state of the environment, the average force vanishes, $\langle V_e \rangle_e = 0$, and then all three quantities decay on the same time scale.

For the purpose of numerical experiment we choose now $V_s = J_z/J$ and $V_e = J_z^2 \big/ J^2$ in order to have a less trivial situation of non-vanishing average force. The initial condition is again $(\vartheta, \varphi)_{s,e} = (\pi/\sqrt{3}, \pi/\sqrt{2})$ and parameters are $J = 200$, $\alpha_s = 30$, $\gamma_s = \pi/7$ and $\alpha_e = 0$, $\gamma_e = \pi/2.1$ and perturbation strength $\delta = 6 \times 10^{-4}$. By choosing the explicitly solvable case $\alpha_e = 0$ we can calculate $\bar{c}_e$ and $\bar{c}_e - \overline{\langle V_e \rangle_e^2}$, say for the simple case of $\pi/2$ rotation, $\gamma_e = \pi/2$, where we obtain

$$\begin{aligned} \bar{c}_e &= \frac{1}{4}\left(1 - y_e^2\right)^2 + \frac{1}{4J}\left(-3y_e^4 + 2y_e^2 + 1\right) + \mathcal{O}(1/J^2) \\ \bar{c}_e - \overline{\langle V_e \rangle_e^2} &= \frac{1}{2J} y_e^2\left(1 - y_e^2\right) + \frac{1}{16J^2}\left(11y_e^4 - 11y_e^2 + 2\right) + \mathcal{O}(1/J^3). \end{aligned} \tag{39}$$

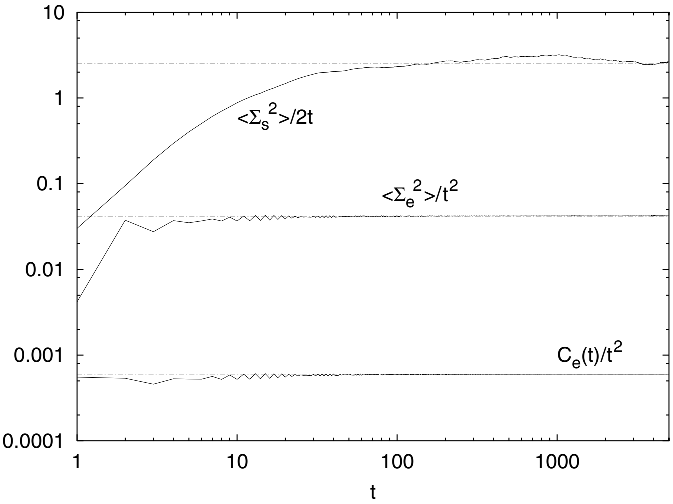

Figure 10. Correlation sums occurring in (38) (solid curves). The top chain line gives the best fit for σ_s and the two lower chain lines give the theoretical time-averaged correlation functions for the environment (39). All for a fast regular regime. See text for details.

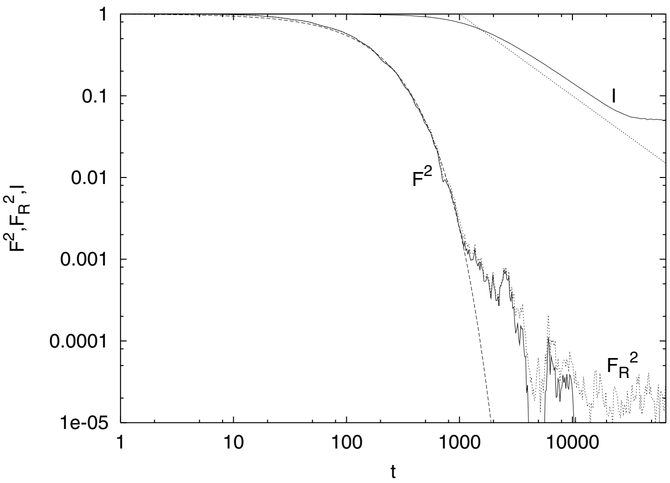

Figure 11. Decay of $F^2(t)$, $F^2_R(t)$ and $I(t)$ for a fast regular environment. The dashed line is an exponential with the exponent given by a product of σ_s and $\overline{c}_e$ (38). The straight dotted line has a slope −1. See text for details.

The values of these coefficients are shown in figure 10 (lower two dotted lines). They agree nicely with numerics for $\gamma_e = \pi/2.1$. In figure 11 we can observe exponential decay of fidelity and reduced fidelity on the same time scale (two curves almost overlap) and decay of purity on a $1/\hbar$ longer time scale. For longer times the purity decay is again algebraic. In figure 12 we show the dependence of decay times on δ. The dependence for purity is quite interesting. If one looks at the time the purity falls to 0.99 one has agreement with linear response (by definition). But if one looks at the purity level 0.37, they do not agree as well, meaning that the

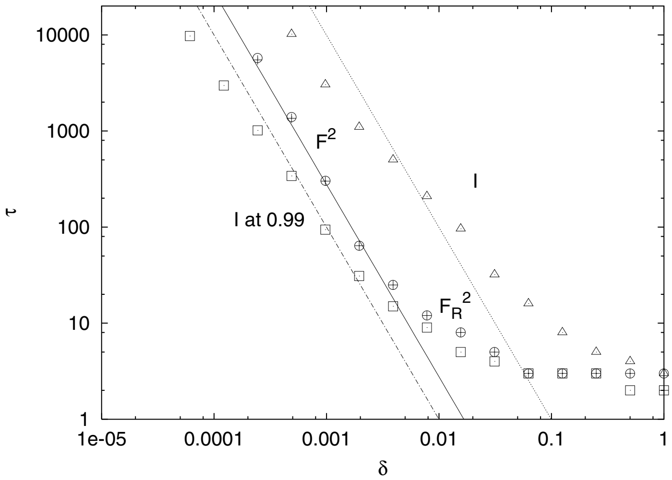

Figure 12. Times τ at which $F^2(t)$ (pluses), $F_R^2(t)$ (circles) and $I(t)$ (triangles) fall to level 0.37, and the times τ when $I(t)$ falls to 0.99 (squares) for varying δ and with a fast regular environment. Symbols are numerics and lines give theoretical dependence of τ. All for $J = 100$.

nature (shape) of the purity decay may change (not only the scale) as one varies δ or $\hbar$. On the other hand, this may also be simply a finite size effect due to finite Hilbert space dimensions.

6. Discussion and conclusions

In this paper, we have analysed the stability of unitary time evolution of composite systems under weak coupling between subsystems. This is a natural (unitary) model for dissipation and decoherence in quantum mechanics where one subsystem plays the role of the central system and the other plays the role of the environment. But this is not the only possible application of the above ideas. One may also be interested in the dynamical effects of weak coupling between two controllable parts of the system, e.g. the atom and the electromagnetic cavity [20].

We have analysed three different quantities that are treated on a similar theoretical footing but which have different physical interpretations. The first two, namely the fidelity and the reduced fidelity, refer to the case of echo dynamics where the forward evolution is generated by the uncoupled system while the backward evolution is generated by the weakly coupled system. The third, namely the purity, refers to the growth of linear entropy or growth of entanglement between the two subsystems during the course of weakly coupled forward evolution. First we have shown a rigorous inequality between the three quantities which may be useful to provide various bounds. Then we have developed a linear response theory which predicts time scales for the three quantities in terms of time-correlation functions of the perturbation in each of the subsystems. Thus we have been able to classify all different behaviours with respect to regularity or chaoticity (as defined here by the mixing property) of each of the subsystems. The general conclusion is again, consistent with [21, 22], that strong chaos stabilizes quantum dynamics with respect to the intersystem coupling, and that strong chaos decreases the rate of entanglement (or linear entropy) growth. In particular, if the characteristic time scales of correlation decay in two subsystems are well separated, then we can integrate over the correlation functions of the fast subsystem and obtain expressions that are independent of the nature of dynamics in the slow subsystem.

However, our results refer strictly to the 'quantum regime' where the strength of perturbation is semiclassically small, namely smaller than the Planck constant, $\delta < \hbar$ [10]. We note that in the opposite, 'classical regime', $\delta > \hbar$, the decay time scales become shorter than the Ehrenfest time $(-\log\hbar)$ and saturate with δ (see e.g. figures 5, 9 and 12). So one obtains the so-called 'perturbation-independent decay' governed by the classical Lyapunov instability [5, 7].

Acknowledgments

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